Noncommutative Deformations & Surface Autoequivalences

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Flops and Derived Categories

Flops are a special type of codimension two surgery: birational maps that are isomorphisms in codimension one. The definition involves a diagram [1] with the $\pi^{\pm}$ small contractions – often π^- is given and we wish to construct ϕ. A threefold flop essentially modifies curves in X^-.

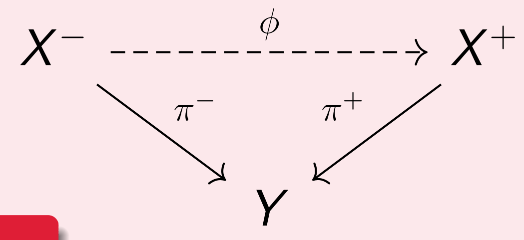

Figure 1: A flop ϕ.

Example: The Atiyah Flop

Y is the cone $\frac{k[u,v,x,y]}{(uv-xy)}$. Blow up the cone point; the exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$. Contract the first $\mathbb{P}^1$ to get X^- and the second to get X^+.

Flops are important in the Minimal Model Program: a theorem of Kawamata says that any two minimal models are connected by a sequence of flops. If X is a variety, the derived category $D(X) := D^b(\mathrm{Coh}(X))$ knows a lot about the birational geometry of X: for example, Bridgeland proved that a flop $X \to X^+$ between smooth projective threefolds induces an equivalence $D(X) \to D(X^+)$. Can one use homological methods to study threefold flops? One invariant of flopping curves, the **contraction algebra**, has been defined by Donovan-Wemyss using noncommutative deformation theory. It subsumes many other invariants, and is conjectured to classify threefold flops completely.

Deformation Theory

Deformation theory is the study of infinitesimal deformations. The infinitesimals are the local Artinian k-algebras with residue field k, e.g. the **dual numbers** $k[\varepsilon] = k[x]/x^2$. A deformation of a scheme X over such a ring Γ is a flat map $\mathcal{X} \to \mathrm{Spec}(\Gamma)$ that pulls back along $\Gamma \to k$ to $X \to \mathrm{Spec}(k)$.

First-order deformations of a plane curve

If $f \in k[x,y]$, then the set of deformations of $\{f = 0\}$ over $k[\varepsilon]$ is $\frac{k[x,y]}{(f,f_x,f_y)}$. For example, picking $\{xy = 0\}$, we get $\frac{k[x,y]}{(xy,y,x)} \cong k$; every deformation is of the form $xy = t\varepsilon$. One can think of this as the 'first-order part' of the family $\mathrm{Spec}\,\frac{k[x,y,t]}{(xy-t)} \to \mathrm{Spec}\,k[t]$ pictured in [2].

Given a scheme X, its **deformation functor** Def_X : **Art** $\to$ **Set** sends Γ to the isoclasses of deformations of X over Γ. It's often (pro)representable, by a local Noetherian k-algebra (e.g. a power series ring). One can do noncommutative or derived deformation theory by modifying the definition of 'infinitesimal': just use noncommutative or dg Artinian algebras. If A is a k-algebra and S is a one-dimensional simple A-module, then the noncommutative derived deformation functor Def_S has prorepresenting object the double Koszul dual $\mathbb{R}\mathrm{End}_{\mathbb{R}\mathrm{End}_A(S)}(k)$.

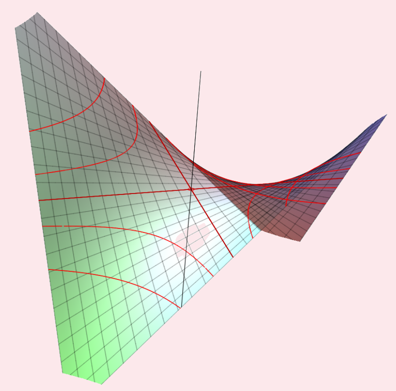

Figure 2: A family over $\mathbb{A}^1$ (black line).

The Contraction Algebra

Fix a contraction $f : X \to X_{\mathrm{con}}$ of not-too-singular threefolds, and pick an irreducible curve $C \cong \mathbb{P}^1$ in the exceptional locus. Does C flop? Using perverse sheaves, Van den Bergh constructs a bundle $\mathcal{V}$ on X and a derived equivalence $D(X) \to D(A)$, where $A = \mathrm{End}_X(\mathcal{V})$. Under this equivalence, $\mathcal{O}_C(-1)$ goes to a simple module S, and the contraction algebra A_{con} is the prorepresenting object for the noncommutative deformation functor Def_S. Importantly, C flops if and only if $\dim_k(A_{\mathrm{con}}) < \infty$.

Examples

The Atiyah flop has contraction algebra k; more generally the Pagoda flop with base $\frac{k[u,v,x,y]}{(uv-(x+y^n)(x-y^n))}$ has contraction algebra $k[t]/t^n$. But A_{con} need not be commutative!

There's a canonical algebra map $g : A \to A_{\mathrm{con}}$; the **noncommutative twist around** A_{con} is the functor $T = \mathbb{R}\mathrm{Hom}_A(\ker(g), -)$. It's an autoequivalence, and if C flops it's the **mutation-mutation autoequivalence** MM. Loosely, one mutates A by perturbing $\mathcal{V}$ to obtain a new ring $B := \mathrm{End}_X(\mu\mathcal{V})$ and a derived equivalence $D(A) \to D(B)$. Mutation is an involution, so mutating again gives an autoequivalence MM of $D(A)$. Wemyss's Homological MMP says that mutations correspond exactly to flops between minimal models: indeed, T globalises to give an autoequivalence of $D(X)$ that's isomorphic to the (inverse of the) Bridgeland-Chen flop-flop functor $D(X) \to D(X^+) \to D(X)$.

A Surface Example

Let's return to the Atiyah flop: cut a 1-curve resolution $\tilde{Y} \to Y$ along $x = y^n$ to obtain a partial resolution $X \to \mathrm{Spec}\,\frac{k[u,v,y]}{(uv-y^{n+1})}$ of an A_n singularity. Do Donovan and Wemyss's methods give an autoequivalence of X? The resolution $\tilde{Y}$ is derived equivalent to the algebra $\tilde{A}$ with quiver presentation [3], and across the equivalence the curve corresponds to S_2, the simple at 2. Cutting yields an algebra A with the same quiver, but where the last two relations are replaced by $at = (sb)^n$ and $ta = (bs)^n$. One can compute $A_{\mathrm{con}} = k$, so A_{con} does not contain much information about surface singularities! What if we consider the derived contraction algebra $A^{\mathrm{der}}_{\mathrm{con}} = \mathbb{R}\mathrm{End}_{\mathbb{R}\mathrm{End}_A(S_2)}(k)$?

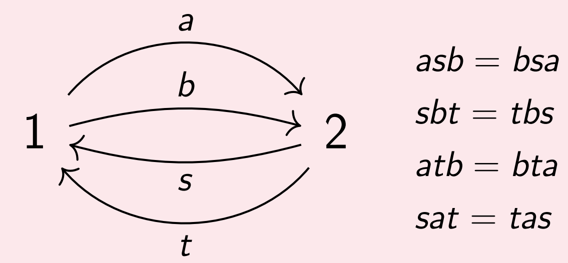

Figure 3: The algebra $\tilde{A}$.

One can identify $A^{\mathrm{der}}_{\mathrm{con}}$ as an A_∞-algebra: it has two generators ζ and η in degrees $-1, -2$ respectively, and only one higher bracket in level $n+1$. In fact, it's an algebra over the subalgebra $k[\eta]$, essentially because MM shifts the simple S_2 by 2; η is obtained from the unit $\mathrm{id} \to MM$. Truncating $A^{\mathrm{der}}_{\mathrm{con}}$ by applying $- \otimes^{\mathbb{L}}_{k[\eta]} k$ recovers the two-term dga defining the mutation-mutation autoequivalence MM; in particular it's not just A_{con}. One can view the noncommutative twist around $A^{\mathrm{der}}_{\mathrm{con}}$ as the infinite composition MM^∞, since $A^{\mathrm{der}}_{\mathrm{con}}$ is the derived completion $\hat{A}_{S_2}$. The same analysis works for threefolds: currently I'm thinking about $A^{\mathrm{der}}_{\mathrm{con}}$ for Pagoda flops. For the Atiyah flop, it's simply $k[\eta]$.

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